Rydberg Atomic Receiver

Lecture Notes on the Hamiltonian of a Ladder-Type Rydberg EIT System

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Chapter 1

Hamiltonian Derivation of a Ladder-Type Rydberg EIT System

Lecture Objective

The objective of this lecture is to derive the effective Hamiltonian of a three-level ladder-type Rydberg Electromagnetically Induced Transparency (EIT) system. The derivation starts from the atomic basis states, introduces the probe and control optical fields, applies the electric-dipole interaction, moves to a rotating frame, and finally obtains the standard EIT Hamiltonian in terms of Rabi frequencies and detunings.

This lecture also explains the physical meaning of the Rabi frequency, the dark state, the two-photon resonance condition, and the connection between the atomic coherence and the observed EIT transparency window.

1-1 Physical Picture of Ladder-Type Rydberg EIT

A ladder-type Rydberg EIT system consists of three atomic states: a ground state, an intermediate excited state, and a highly excited Rydberg state. The lower transition is driven by a weak probe laser, while the upper transition is driven by a stronger control laser. The two optical fields create quantum interference between excitation pathways. Under the two-photon resonance condition, this interference suppresses population in the lossy intermediate state and produces a transparency window for the probe field.

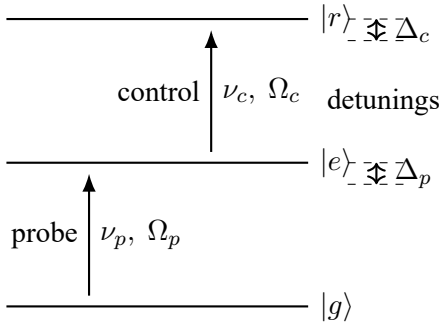

Figure 1-1: Basic ladder-type Rydberg EIT configuration. The probe field couples $|g\rangle \leftrightarrow |e\rangle$, while the control field couples $|e\rangle \leftrightarrow |r\rangle$. The detunings Δ_p and Δ_c represent the resonance offsets of the probe and control fields, respectively.

1-2 Choice of Atomic Basis States

For a ladder system, we consider the following three atomic basis states:

$$\{|g\rangle\,,\ |e\rangle\,,\ |r\rangle\}\,. \tag{1-1}$$

Here, $|g\rangle$ is the ground state, $|e\rangle$ is the intermediate excited state, and $|r\rangle$ is the Rydberg state. The allowed optical transition pathway is

$$|g\rangle \leftrightarrow |e\rangle \leftrightarrow |r\rangle\,. \tag{1-2}$$

The probe field couples the lower transition $|g\rangle \leftrightarrow |e\rangle$, whereas the control field couples the upper transition $|e\rangle \leftrightarrow |r\rangle$. In the simple three-level model used here, the direct optical transition $|g\rangle \leftrightarrow |r\rangle$ is not included.

1-3 Field-Free Atomic Hamiltonian

In the absence of external optical fields, the Hamiltonian contains only the internal energies of the three atomic states. We denote these energies by

$$\varepsilon_g, \qquad \varepsilon_e, \qquad \varepsilon_r. \tag{1-3}$$

The field-free atomic Hamiltonian is therefore

$$\hat{H}_0 = \varepsilon_g\,|g\rangle\,\langle g| + \varepsilon_e\,|e\rangle\,\langle e| + \varepsilon_r\,|r\rangle\,\langle r|\,. \tag{1-4}$$

In the ordered basis $\{|g\rangle\,,|e\rangle\,,|r\rangle\}$, this Hamiltonian has the matrix representation

$$\mathbf{H}_0 = \begin{pmatrix} \varepsilon_g & 0 & 0 \\ 0 & \varepsilon_e & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}. \tag{1-5}$$

The diagonal elements represent the energies of the atomic states. The off-diagonal elements are zero because no external field has yet coupled the states.

1-4 Optical Fields in the Ladder Configuration

Two optical fields are used in the ladder EIT configuration. The first field is the weak probe field, which drives the lower transition

$$|g\rangle \leftrightarrow |e\rangle . \tag{1-6}$$

Its scalar electric-field component along the relevant dipole direction is written as

$$\mathcal{E}_p(t) = \tilde{\mathcal{E}}_p e^{-i\nu_p t} + \tilde{\mathcal{E}}_p^* e^{i\nu_p t}, \tag{1-7}$$

where ν_p is the probe-field frequency and $\tilde{\mathcal{E}}_p$ is the slowly varying complex field amplitude.

The second field is the control field, which drives the upper transition

$$|e\rangle \leftrightarrow |r\rangle . \tag{1-8}$$

Its electric-field component is written as

$$\mathcal{E}_c(t) = \tilde{\mathcal{E}}_c e^{-i\nu_c t} + \tilde{\mathcal{E}}_c^* e^{i\nu_c t}, \tag{1-9}$$

where ν_c is the control-field frequency and $\tilde{\mathcal{E}}_c$ is its slowly varying complex amplitude.

1-5 Electric-Dipole Atom-Field Interaction

In the electric-dipole approximation, the atom-field interaction is

$$\hat{H}_{\text{int}} = -\hat{\boldsymbol{\mu}} \cdot \mathbf{E}, \tag{1-10}$$

where $\hat{\boldsymbol{\mu}}$ is the electric-dipole operator and $\mathbf{E}$ is the electric-field vector. After projecting the field along the relevant transition dipole direction, the coupling matrix elements are

$$H_{ge}(t) = -\mu_{ge}\mathcal{E}_p(t), \tag{1-11}$$

for the probe transition and

$$H_{er}(t) = -\mu_{er}\mathcal{E}_c(t), \tag{1-12}$$

for the control transition. Here, μ_{ge} is the transition dipole matrix element between $|g\rangle$ and $|e\rangle$, and μ_{er} is the transition dipole matrix element between $|e\rangle$ and $|r\rangle$.

In this three-level approximation, the direct ground-to-Rydberg coupling is set to zero:

$$H_{gr} = H_{rg} = 0. \tag{1-13}$$

1-6 Full Hamiltonian in the Laboratory Frame

By combining the field-free Hamiltonian and the dipole interaction, the full laboratory-frame Hamiltonian is obtained. In the ordered basis $\{|g\rangle, |e\rangle, |r\rangle\}$, it is

$$\mathbf{H}_{\text{lab}}(t) = \begin{pmatrix} \varepsilon_g & -\mu_{ge}\mathcal{E}_p(t) & 0 \\ -\mu_{ge}^*\mathcal{E}_p^*(t) & \varepsilon_e & -\mu_{er}\mathcal{E}_c(t) \\ 0 & -\mu_{er}^*\mathcal{E}_c^*(t) & \varepsilon_r \end{pmatrix}. \tag{1-14}$$

The diagonal elements are the bare atomic energies. The off-diagonal element $-\mu_{ge}\mathcal{E}_p(t)$ is the probe-induced coupling between $|g\rangle$ and $|e\rangle$, while $-\mu_{er}\mathcal{E}_c(t)$ is the control-induced coupling between $|e\rangle$ and $|r\rangle$.

1-7 Transition Frequencies and Detunings

The transition frequency between the ground state and the intermediate excited state is

$$\omega_{ge} = \frac{\varepsilon_e - \varepsilon_g}{\hbar}. \tag{1-15}$$

The transition frequency between the intermediate excited state and the Rydberg state is

$$\omega_{er} = \frac{\varepsilon_r - \varepsilon_e}{\hbar}. \tag{1-16}$$

The two-photon transition frequency between $|g\rangle$ and $|r\rangle$ is therefore

$$\omega_{gr} = \frac{\varepsilon_r - \varepsilon_g}{\hbar} = \omega_{ge} + \omega_{er}. \tag{1-17}$$

The probe detuning is defined as

$$\Delta_p = \nu_p - \omega_{ge}, \tag{1-18}$$

while the control detuning is

$$\Delta_c = \nu_c - \omega_{er}. \tag{1-19}$$

For a ladder system, the two-photon detuning is

$$\delta = \Delta_p + \Delta_c. \tag{1-20}$$

The condition $\delta = 0$ is the two-photon resonance condition.

1-8 Rotating Frame and Rotating-Wave Approximation

The laboratory-frame Hamiltonian in (1-14) contains fast oscillations at optical frequencies. To focus on the slowly varying atomic dynamics, we transform to a rotating frame and apply the rotating-wave approximation (RWA). In this rotating frame, the effective energies become

$$\varepsilon_g \to \varepsilon_g, \tag{1-21}$$
$$\varepsilon_e \to \varepsilon_e - \hbar\nu_p, \tag{1-22}$$
$$\varepsilon_r \to \varepsilon_r - \hbar(\nu_p + \nu_c). \tag{1-23}$$

The last expression appears because reaching $|r\rangle$ from $|g\rangle$ requires a two-photon excitation: one probe photon and one control photon.

After removing the rapidly oscillating counter-rotating terms, the rotating-frame Hamiltonian becomes

$$\mathbf{H}_R = \begin{pmatrix} \varepsilon_g & -\mu_{ge}\tilde{\mathcal{E}}_p & 0 \\ -\mu_{ge}^*\tilde{\mathcal{E}}_p^* & \varepsilon_e - \hbar\nu_p & -\mu_{er}\tilde{\mathcal{E}}_c \\ 0 & -\mu_{er}^*\tilde{\mathcal{E}}_c^* & \varepsilon_r - \hbar(\nu_p + \nu_c) \end{pmatrix}. \tag{1-24}$$

1-9 Standard Hamiltonian Using Rabi Frequencies

The optical couplings are usually expressed in terms of Rabi frequencies. With the field-amplitude convention used in this lecture, we define

$$\Omega_p = \frac{\mu_{ge}\tilde{\mathcal{E}}_p}{\hbar}, \tag{1-25}$$

and

$$\Omega_c = \frac{\mu_{er}\tilde{\mathcal{E}}_c}{\hbar}. \tag{1-26}$$

The quantity Ω_p describes the strength of the probe coupling on the $|g\rangle \leftrightarrow |e\rangle$ transition, while Ω_c describes the strength of the control coupling on the $|e\rangle \leftrightarrow |r\rangle$ transition. Some texts include a factor of $1/2$ in the off-diagonal Hamiltonian terms. That difference comes from the chosen convention for defining the complex field amplitude and the Rabi frequency.

Choosing the zero of energy at $|g\rangle$, or equivalently subtracting $\varepsilon_g\mathbf{I}$ from the Hamiltonian, gives the standard ladder EIT Hamiltonian

$$\mathbf{H}_{R,\text{ladder}} = \hbar \begin{pmatrix} 0 & -\Omega_p & 0 \\ -\Omega_p^* & -\Delta_p & -\Omega_c \\ 0 & -\Omega_c^* & -\delta \end{pmatrix}. \tag{1-27}$$

This is the central Hamiltonian used for the ideal coherent description of a three-level ladder-type Rydberg EIT system.

1-10 Physical Meaning of the Rabi Frequency

The Rabi frequency measures the atom-field coupling strength. For a two-level transition, a resonant optical field can drive coherent population oscillations between two states:

$$|g\rangle \rightarrow |e\rangle \rightarrow |g\rangle \rightarrow \cdots. \tag{1-28}$$

The rate of this coherent exchange is controlled by the Rabi frequency. A larger Ω corresponds to stronger atom-field coupling and faster coherent oscillation.

The reason Rabi frequencies are useful is that they convert the coupling energy into a frequency scale:

$$-\mu_{ge}\tilde{\mathcal{E}}_p = -\hbar\Omega_p, \qquad -\mu_{er}\tilde{\mathcal{E}}_c = -\hbar\Omega_c. \tag{1-29}$$

Therefore, all entries inside the matrix in (1-27) have units of angular frequency, while the factor $\hbar$ appears outside the matrix.

The Rabi frequency is proportional to the electric-field amplitude:

$$\Omega \propto \tilde{\mathcal{E}}. \tag{1-30}$$

Since optical intensity is proportional to the squared field amplitude, we also have

$$\tilde{\mathcal{E}} \propto \sqrt{I}, \qquad \Omega \propto \sqrt{I}. \tag{1-31}$$

Thus, increasing the optical power increases the Rabi frequency. In EIT, the control-field Rabi frequency Ω_c is especially important because it controls the width and visibility of the transparency window.

1-11 Dark State in the Ladder System

The key physical mechanism of EIT is the formation of a dark state. Under the two-photon resonance condition

$$\delta = 0, \tag{1-32}$$

the system supports a coherent superposition that does not contain the lossy intermediate state $|e\rangle$. For real Rabi frequencies, this dark state can be written as

$$|D\rangle = \frac{\Omega_c |g\rangle - \Omega_p |r\rangle}{\sqrt{|\Omega_p|^2 + |\Omega_c|^2}}. \tag{1-33}$$

For complex Rabi frequencies, the relative phases must be kept consistently. A convenient phase convention gives

$$|D\rangle = \frac{\Omega_c |g\rangle - \Omega_p^* |r\rangle}{\sqrt{|\Omega_p|^2 + |\Omega_c|^2}}. \tag{1-34}$$

The important point is that $|D\rangle$ has no $|e\rangle$ component. Therefore, the probe field cannot efficiently populate the intermediate excited state, and absorption is strongly suppressed.

In the common weak-probe regime,

$$|\Omega_p| \ll |\Omega_c|, \tag{1-35}$$

the dark state is approximately

$$|D\rangle \approx |g\rangle . \tag{1-36}$$

This explains why the medium can remain mostly in the ground state while still becoming transparent to the probe field.

1-12 Effect of Nonzero Two-Photon Detuning

If the two-photon detuning is not zero,

$$\delta \neq 0, \tag{1-37}$$

the exact destructive interference condition is no longer perfectly satisfied. The relative phase between the $|g\rangle$ and $|r\rangle$ components changes in time, and the dark state becomes imperfect. A useful intuitive form is

$$|D_\delta(t)\rangle = \frac{\Omega_c |g\rangle - e^{i\delta t}\Omega_p |r\rangle}{\sqrt{|\Omega_p|^2 + |\Omega_c|^2}}. \tag{1-38}$$

For small δ, EIT can still be observed but the transparency is reduced. For large δ, the destructive interference is lost and probe absorption reappears. Therefore, the main condition for a clear EIT window is

$$\delta = \Delta_p + \Delta_c \simeq 0. \tag{1-39}$$

1-13 Including Decay and Decoherence

The coherent Hamiltonian in (1-27) is the first step. In a realistic EIT medium, the intermediate and Rydberg states can decay or dephase. These effects can be included using a density matrix $\boldsymbol{\rho}$ and a master equation of the form

$$\frac{d\boldsymbol{\rho}}{dt} = -\frac{i}{\hbar}\left[\mathbf{H}_{R,\text{ladder}}, \boldsymbol{\rho}\right] + \mathcal{L}_{\text{decay}}(\boldsymbol{\rho}), \tag{1-40}$$

where $\mathcal{L}_{\text{decay}}(\boldsymbol{\rho})$ represents spontaneous decay and decoherence. Typical parameters include the intermediate-state decay rate γ_e, the Rydberg-state decay rate γ_r, and the ground-Rydberg coherence decay rate γ_{gr}.

A useful estimate for the coherence time is

$$\tau_{\text{coh}} \sim \frac{1}{\gamma_{gr}}. \tag{1-41}$$

Longer coherence time generally improves the sharpness of the EIT feature.

1-14 Probe Absorption and the EIT Window

In an EIT measurement, the probe field is the optical readout channel. The absorption of the probe is related to the imaginary part of the atomic coherence on the lower transition:

$$\alpha_p \propto \text{Im}\left[\rho_{ge}\right]. \tag{1-42}$$

Here, ρ_{ge} is the coherence between $|g\rangle$ and $|e\rangle$. The probe transmission through a medium of length L can be modeled as

$$T = e^{-\alpha_p L}. \tag{1-43}$$

When $\delta \simeq 0$, the dark state suppresses $\text{Im}[\rho_{ge}]$. As a result, α_p decreases and the probe transmission increases. This transmission peak is the EIT transparency window.

1-15 Example: Rydberg EIT in Rubidium

A typical ladder-type Rydberg EIT system in rubidium can be described using

$$|g\rangle = 5S_{1/2}, \tag{1-44}$$

$$|e\rangle = 5P_{3/2}, \tag{1-45}$$

$$|r\rangle = nS \quad \text{or} \quad nD. \tag{1-46}$$

The probe field drives

$$5S_{1/2} \leftrightarrow 5P_{3/2}, \tag{1-47}$$

and the control field drives

$$5P_{3/2} \leftrightarrow nS \quad \text{or} \quad 5P_{3/2} \leftrightarrow nD. \tag{1-48}$$

The EIT condition is again

$$\delta = 0. \tag{1-49}$$

Under this condition, a dark state is formed between the ground state and the Rydberg state, reducing probe absorption and producing the observable EIT window.

1-16 Step-by-Step Design Procedure

Figure 1-2 summarizes the main steps for designing and analyzing a ladder-type Rydberg EIT system.

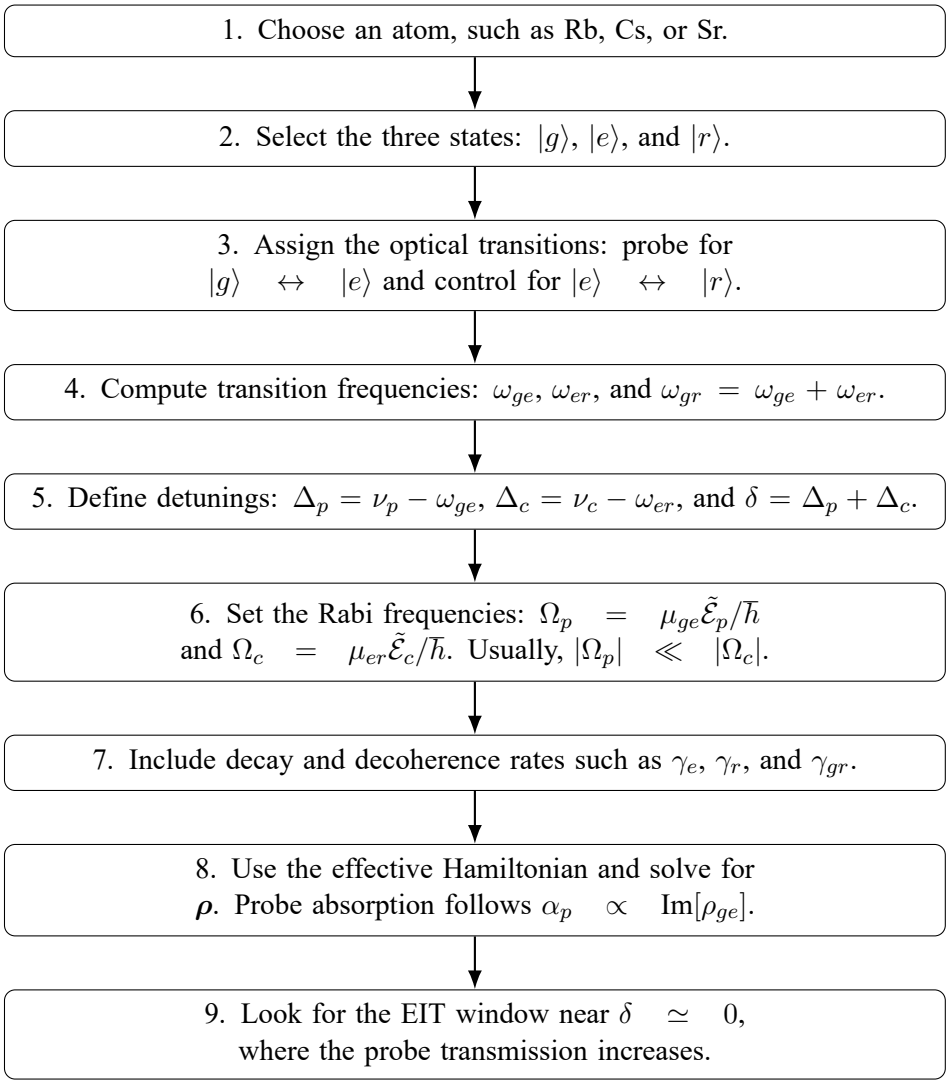
1. Choose an atom, such as Rb, Cs, or Sr.
2. Select the three states: $|g\rangle$, $|e\rangle$, and $|r\rangle$.
3. Assign the optical transitions: probe for $|g\rangle \leftrightarrow |e\rangle$ and control for $|e\rangle \leftrightarrow |r\rangle$.
4. Compute transition frequencies: ω_{ge}, ω_{er}, and $\omega_{gr} = \omega_{ge} + \omega_{er}$.
5. Define detunings: $\Delta_p = \nu_p - \omega_{ge}$, $\Delta_c = \nu_c - \omega_{er}$, and $\delta = \Delta_p + \Delta_c$.
6. Set the Rabi frequencies: $\Omega_p = \mu_{ge}\tilde{\mathcal{E}}_p/\hbar$ and $\Omega_c = \mu_{er}\tilde{\mathcal{E}}_c/\hbar$. Usually, $|\Omega_p| \ll |\Omega_c|$.
7. Include decay and decoherence rates such as γ_e, γ_r, and γ_{gr}.
8. Use the effective Hamiltonian and solve for $\boldsymbol{\rho}$. Probe absorption follows $\alpha_p \propto \mathrm{Im}[\rho_{ge}]$.
9. Look for the EIT window near $\delta \simeq 0$, where the probe transmission increases.

Figure 1-2: Step-by-step procedure for designing and analyzing a ladder-type Rydberg EIT system.

1-17 Summary

The main result of this lecture is the rotating-frame Hamiltonian for a ladder-type Rydberg EIT system:

$$\mathbf{H}_{R,\text{ladder}} = \hbar \begin{pmatrix} 0 & -\Omega_p & 0 \\ -\Omega_p^* & -\Delta_p & -\Omega_c \\ 0 & -\Omega_c^* & -\delta \end{pmatrix}. \tag{1-50}$$

The EIT condition is

$$\delta = \Delta_p + \Delta_c = 0. \tag{1-51}$$

Under this condition, the dark state suppresses the excitation of the intermediate state and reduces probe absorption. The physical origin of the EIT window is therefore destructive quantum interference in the ladder system.

Table 1-1: Key quantities in the ladder-type Rydberg EIT model.

Symbol	Meaning
$\|g\rangle$	Ground atomic state
$\|e\rangle$	Intermediate excited state
$\|r\rangle$	Rydberg state
ν_p	Probe-laser frequency
ν_c	Control-laser frequency
Ω_p	Probe Rabi frequency
Ω_c	Control Rabi frequency
Δ_p	Probe detuning
Δ_c	Control detuning
δ	Two-photon detuning, $\delta = \Delta_p + \Delta_c$
ρ_{ge}	Atomic coherence associated with probe absorption
T	Probe transmission through the atomic medium